

A CONSTRUCTION OF THE STABLE WEB

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ABSTRACT. We provide a process on the space of coalescing cadlag stable paths and show convergence in the appropriate topology for coalescing stable random walks on the integer lattice.

MSC 2010: 82B44, 82D30, 60K37

KEYWORDS: Stable processes, weak convergence, Hausdorff topology

1. Introduction

A system of coalescing Brownian motions starting at “every” point in \mathbb{R} and evolving independently before coalescence was first introduced by Arratia in [A1, A2]. This system have being studied by several authors and motivated the question about the existence of a system of coalescing Brownian Motion starting at “every” point in the space-time plane \mathbb{R}^2 . Such an object is called the Brownian Web and was introduced by Fontes, Isopi, Newman and Ravishankar in [FINR]. In the same paper they prove weak convergence to the Brownian Web under diffusive scaling of the system of simple symmetric one-dimensional coalescing random walks starting on each point in the space-time lattice \mathbb{Z}^2 . Later Newman, Ravishankar and Sun [NRS] proved an invariance principle related to the Brownian Web, they established the convergence to Brownian web for systems of one-dimensional coalescing random walks under finite absolute fifth moment of the transition probability (allowing for crossing of paths unlike the nearest neighbor walks).

More recently, Evans, Morris and Sen [EMS] studied a system of coalescing α -stable processes, $\alpha > 1$, starting at every point in \mathbb{R} . As Arratia [A2] did for Brownian Motion, they proved that the system of α -stable processes are locally finite for every time $t > 0$. And based on this, our main motivation here is to build a stable version of the Brownian Web or simply a “Stable Web” and also prove an invariance principle for it.

In this note we make a first step at defining the stable web. In subsequent work Hao Xue and the last two authors will generalize the domain of applicability and show that the object defined is equivalent to an object with a more general “smoother” topology. Our objective here is simply to define a reasonable metric on the space of collections of cadlag paths that gives convergence to the “stable web” for suitably normalized coalescing random walks.

2. The Age Process

In the following we have for $n \in \mathbb{Z}_+$ $D_n = \mathbb{Z}/2^n$.

We follow [EMS] and consider systems of coalescing identically distributed stable processes $\overline{X}^{D_n} = \{X^{n,x} = (X_t^{n,x})_{t \geq 0} : x \in D_n\}$ with stable index $\alpha \in (1, 2)$ such that $X_0^{n,x} = x = i2^{-n}$ for integers i and n . The stable processes evolve independently until coalescence. The rules of precedence will be arbitrary for points in D_n/D_{n-1} but lower order points will have coalescence precedence so that $\overline{X}^{D_n} \subset \overline{X}^{D_{n+1}}$ for each $n \geq 1$. For $x \in D = \cup_{n \geq 1} D_n$, take n such that $x \in D_n/D_{n-1}$ and simply write $X^{x,n} = X^x$. Moreover we denote $\overline{X} = \cup_{n \geq 1} \overline{X}^{D_n}$

and for every $t > 0$, $\overline{X}_t^{D_n} = \cup_{x \in D_n} X_t^x$ and $\overline{X}_t = \cup_{x \in D} X_t^x$ which are the time level sets associated to the set valued processes \overline{X}^{D_n} and \overline{X} respectively. Arguing as in [BG] we have:

Proposition 1. *There exists $K < \infty$ so that $\forall n \geq 1$ the density of the process, \overline{X}^{D_n} at time $t < \infty$ $D(n, t)$ given by the a.s. limit of*

$$\lim_{M \rightarrow \infty} \frac{1}{2M} \#(\overline{X}_t^{D_n} \cap [-M, M])$$

satisfies $D(n, t) \leq \frac{K}{t^\alpha}$

It follows from the fact that 0 is regular for the stable processes that if we choose a $x \notin D$ and start a stable process at x at time 0 which will coalesce with processes starting at D (given precedence to the latter) then $P(X_t^x \in \overline{X}_t \forall t > 0) = 1$. For any $x \in \mathbb{R}$ we can indeed unambiguously define a stable process $(X_t^x)_{t \geq 0}$ which has the same distribution as a stable process starting at x and such that $t > 0$, $X_t^x \in \overline{X}_t$. So we can think of the above process as a collection of coalescing stable processes starting “on” \mathbb{R} . For any set $E \subset \mathbb{R}$ we denote

$\overline{X}^E = \{X^x : x \in E\}$. It follows that if for any nested collection of “translation invariant” points V_n with $V = \cup_{n \geq 1} V_n$ dense we have (with the coalescence rules with \overline{X}_t as above) that $\overline{X}_t^{V_n} \subset \overline{X}_t \forall t > 0$ with probability 1. But equally we can show a.s that $\overline{X}_t \subset \overline{X}_t^V$. Furthermore for any strictly positive c we can choose V_0 to be points spaced c apart and containing 0 and V_n obtained from V_{n-1} by adding the midpoints between neighbours. Then we have that

$$\overline{X}_t^{V_n} \xrightarrow{D} \overline{X}_t, \quad \overline{X}_t^{D_n} \xrightarrow{D} \overline{X}_t^V \quad \text{and} \quad \frac{1}{c} \overline{X}_{c^\alpha t}^{V_n} \stackrel{D}{=} \overline{X}_t^n,$$

together this yields.

Proposition 2. *For the process \overline{X}_t the density at time t is equal to k/t^α for some k depending on our choice of the stable process.*

To construct the stable web we now consider coalescing stable processes starting at times $t \in D$.

The first step is for $D_0 = \mathbb{Z}^1$. We define the stable coalescing processes starting at $\{i\} \times \mathbb{R}$. For $t \in (i, i+1]$ let this process be \overline{X}_t^i . At time $t = i+1$, \overline{X}_{i+1}^i will be a countable collection of points on $\{i+1\} \times \mathbb{R}$. As such they can be continued on interval $[i+1, i+2]$ so that $\overline{X}_s^i \subset \overline{X}_s^{i+1}$ for every $s \in [i+1, i+2]$ with probability 1. Continuing we have $\forall i < j$ $\overline{X}_s^i \subset \overline{X}_s^j \forall s > j$ a.s.

We now proceed in an analagous manner adding in stable coalescing processes at times D_n/D_{n-1} to obtain a collection of processes $\{\overline{X}_t^d\}_{t > d}$ for $d \in D$ with the property that a.s. for every $d < d'$, $t > d'$ we have that $\overline{X}_t^d \subset \overline{X}_t^{d'}$. We use the notation $X^{d,d'} = (X_t^{d,d'})_{t \geq d}$ to denote the stable process beginning at (dyadic) time d at spatial (dyadic) point d' .

We now define the *age* of a process (or path) $(\gamma(s))_{s > d}$ of \overline{X}^d : the age of $(\gamma(s), s)$ is simply $s - \inf\{d' < d : \gamma(s) \in \overline{X}_s^{d'}\}$. So the age of $(\gamma(s), s)$ increases continuously at rate 1 but then jumps when the path coalesces with an older path.

From this we can for each ε consider the ε -processes or ε -paths consisting of paths whose age is greater than or equal to ε : For every $d, d' \in D$, for the path with $Y = X_d^{d,d'} = d'$, we let $A_Y(\delta) = \inf\{s > d : (X_s^{d,d'}, s) \text{ has age } \geq \delta\}$, then the path $(Y_t)_{t \geq d}$ is replaced by

$(Y_t)_{t \geq A_Y(\varepsilon)}$ which we call an δ -process. Since we will be interested in distinct δ -paths, we can regard paths for which the age at $A(\delta)$ strictly exceeds δ (i.e. the age is raised as the result of a coalescence with an “older” path) as being suppressed. Thus in a certain way we are imposing that older paths have coalescence precedence over the others. We leave the reader to check that there is no infinite regress and not all δ -paths are suppressed and the union of all points touched by unsuppressed δ -paths at a time t gives all the points of age greater than δ at time t . We similarly note that almost surely no path can achieve age precisely δ via a coalescence. We write Φ_δ for this operation on the set of aged paths.

In general we say a collection of aged paths is a set of triples (I, γ, a) (where I is an open interval, $\gamma : I \rightarrow \mathbb{R}$ is a cadlag function and $a : I \rightarrow \mathbb{R}_+$ is a cadlag function satisfying for $s' < s$, $a(s) \geq a(s') + (s - s')$). When $I = (b, \infty)$ we simply write $(I, \gamma, a) = (b, \gamma, a)$.

We put $\mathcal{X} = \cup_{d \in D} \{([d, \infty), Y, a_Y) : Y \in \overline{X}^d\}$ and $\mathcal{X}_\delta = \Phi_\delta(\mathcal{X})$ as our systems of coalescing stable aged processes and coalescing δ -processes. Given a countable dense collection of space time points $\{(x_i, t_i)\}_{i=1}^\infty$ we can also define for stable aged processes beginning at these space time points which are subordinate (or contained in) to our “stable web” aged process.

In our system of paths, as already noted, a δ -path $\gamma : [c, b] \rightarrow \mathbb{R}$ is a path so that $(\gamma(c), c)$ has age at least δ . If $(\gamma(c), c)$ has age exactly δ we say it is δ -fixed path.

Finally note that Proposition 2 yields the following corollary:

Corollary 3. *The density of points of \mathcal{X} with age in the interval $(a, a + \varepsilon)$ at a given time t is equal to $k/a^{1/\alpha} - k/(a + \varepsilon)^{1/\alpha}$.*

We will also need the following result which follows from the fact that the density of processes of age at least M at a particular time tends to zero as M becomes large.

Lemma 4. *Given $\varepsilon > 0$ and $N < \infty$, there exists $M = M(\varepsilon, N) < \infty$ so that outside probability ε every path of \mathcal{X} that intersects space time square $[-N, N]^2$ has age less than M at time N .*

Proof. This simply follows from the fact that by Proposition 2, the density of coalescing processes, started at time $-(N + M)$ has density $k/(2N + M)^{1/\alpha}$ at time N . So the chance that one such process is in spatial interval $[-N, N]$ at time N is less than $2kN/(2N + M)^{1/\alpha}$. Let $c(N) > 0$ be the infimum of the conditional probability a path be in $[-N, N]$ at time N given that it hits $[-N, N]^2$. The probability that the event of interest occurs is bounded above by $2kN/c(N)(2N + M)^{1/\alpha}$ which will be less than ε for large enough M depending on N and ε . \square

3. Topology

First recall (see e.g. [EK]) the definition of the δ -modulus of continuity for a cadlag path $\gamma : [c, d] \rightarrow \mathbb{R}$:

$$\omega(\delta, \gamma, [e, f]) = \inf_{t_i - t_{i-1} \geq \delta} \sup_i \sup_{s, t \in [t_{i-1}, t_i]} |\gamma(t) - \gamma(s)|$$

for $[e, f] \subset [c, d]$. This quantity is important for determining the compactness of sets of cadlag paths.

We use the metric d_1 between two cadlag paths $\gamma_1 : [a, b] \rightarrow \mathbb{R}$ and $\gamma_2 : [c, d] \rightarrow \mathbb{R}$ (typically but not always we will have $b = d = \infty$) where

$$d_1(\gamma_1, \gamma_2) = |\tanh(a) - \tanh(c)| + \inf_{g: [a, b] \rightarrow [c, d]} \left[\sup_{a \leq t \leq b} e^{-|t|} |((\gamma_1(t), t) - (\gamma_2(g(t)), g(t))) \wedge 1) + \sup_{a < s < b} e^{-|s|} |g'(s) - 1| \right].$$

where the infimum is over continuous piecewise differentiable bijections g . This amounts to a compactification of space-time as in [FINR]. When dealing with finite space time rectangles $[S, T] \times [A, B]$, we will use the equivalent metric

$$d(\gamma_1, \gamma_2) = |a - c| + \inf_{g: [a, b] \rightarrow [c, d]} \left[\sup_{a \leq t \leq b} ((\gamma_1(t), t) - (\gamma_2(g(t)), g(t))) \wedge 1) + \sup_{a < s < b} |g'(s) - 1| \right].$$

In dealing with aged paths defined over finite intervals

$$(\gamma_i(t), a_i(t)) : [c_i, b_i] \rightarrow \mathbb{R} \times (0, \infty)$$

for $i = 1, 2$, we simply take

$$d((\gamma_1, a_1), (\gamma_2, a_2)) = d(\gamma_1, \gamma_2) \vee d(a_1, a_2).$$

and similarly for d_1 . In the following, when speaking of distance between aged paths $(\gamma_1, a_1), (\gamma_2, a_2)$, we will abuse notation and write $d_1(\gamma_1, \gamma_2)$. It follows immediately,

A similar topology on cadlag functions was introduced in the recent work of Etheridge et al [EFS].

Lemma 5. *For a cadlag function $f : [0, T] \rightarrow \mathbb{R}$ let f^η be its restriction to $[\eta, T]$ (for $\eta > 0$). $\forall \sigma > 0$, there exists η_0 so that $d(f, f^\eta) < \sigma$ for every $0 \leq \eta \leq \eta_0$.*

Proof. Take h to be such that $\sup_{s \leq h} |f(s) - f(0)| \leq \sigma/10$. Now (for $h > \eta > 0$) define path $g : [0, T] \rightarrow [\eta, T]$ by

$$g(s) = s \text{ for } s \in [h, T]; \quad g(\cdot) \text{ is linear bijection } [0, h] \rightarrow [\eta, h].$$

Then this shows that $d(f, f^\eta) < 2\sigma/10 + \eta/h + \eta$ and so the result follows. \square

The above argument in fact yields

Lemma 6. *For any $h > \eta$,*

$$d(f^\eta, f) \leq \eta + \frac{\eta}{h} + 2 \sup_{0 \leq s \leq h} |f(0) - f(s)|.$$

If we are dealing with aged paths

$$d((f^\eta, a^\eta), (f, a)) \leq \eta + \frac{\eta}{h} + 2 \left(\sup_{0 \leq s \leq h} |f(0) - f(s)| \vee a(h) - a(0) \right).$$

Corollary 7. *For a stable process $(X(t) : t \geq 0)$, $\forall \sigma > 0$ there exists $\eta_0 > 0$ so that $\forall T > 0$,*

$$P(\forall \eta \leq \eta_0, d(X, X^\eta) < \sigma) > 1 - \frac{\sigma^2}{10^6},$$

where we consider the restrictions of X and X^η to $[0, T]$ and $[\eta, T]$ respectively.

Similarly, if for process X and time interval I within its domain of definition, we write X^I as the process with time restricted to I , then we have

Corollary 8. *For a stable process $(X(t) : t \geq 0)$, $\forall \sigma > 0$, $\varepsilon > 0$ there exists $\eta_0 > 0$ so that $\forall t > \varepsilon$ and $0 \leq \eta_1, \eta_2 \leq \eta_0$,*

$$d(X^{[0,t]}, X^{[\eta_1, t+\eta_2]}) < \sigma$$

outside probability $\frac{\sigma^2}{10^6}$

We will examine the systems of aged paths \mathcal{X} and \mathcal{X}_δ considering them as random elements of a proper path space which we now define in a natural way based on our previous considerations. Let G be the space of aged paths: $\{(b, \gamma, a)\}$ where γ and a are cadlag functions defined on (b, ∞) and a satisfies $a(s) \geq a(s') + (s - s')$ for every $s > s' > b$. For each time-space rectangle $[S, T] \times [A, B]$, we define the ‘‘restriction’’ of an aged path $\{(b, \gamma, a)\}$ to $[S, T] \times [A, B]$ as the triple

$$\{(b^{[S,T] \times [A,B]}, \gamma^{[S,T] \times [A,B]}, a^{[S,T] \times [A,B]})\}$$

where:

$$b^{[S,T] \times [A,B]} = \inf\{r > b \vee S : \gamma(r) \in [A, B]\}$$

and if $b > T$,

$$(\gamma^{[S,T] \times [A,B]}, a^{[S,T] \times [A,B]}) = \emptyset,$$

otherwise, for $r > b^{[S,T] \times [A,B]}$,

$$\gamma^{[S,T] \times [A,B]}(r) = \gamma(r)$$

and define $a^{[S,T] \times [A,B]}(\cdot)$ analogously.

For $S = A = -N$, $B = T = N$, we denote by Ψ_N the map that associates to a path in G its restriction. We note that this is not a continuous operator for the given metric between paths.

We have the semimetric $\rho_{[S,T] \times [A,B]}$ on G defined by the d-distance between (b, γ, a) and (b', γ', a') restricted to interval $[S, T] \times [A, B]$.

We denote the composition $\Psi_N \circ \Phi_{2^{-N}}$ by Π_N (recall Φ_δ is defined at the end of Section 2). We now consider the metric between aged paths (which by abuse of notation we also denote as ρ) by

$$\rho((b, \gamma, a), (b', \gamma', a')) = \sum_{N=1}^{\infty} 2^{-N} \min\{1, \rho_{[-N, N]^2}(\Pi_N(b, \gamma, a), \Pi_N(b', \gamma', a'))\}.$$

The metric is artificial in that it privileges certain cutoffs and rectangles, whereas the spatial or temporal integer values are not special for stable processes and the ages 2^{-N} are not significant for our coalescing system. However this is a positive in our approach as we will be able to argue that the lack of continuity of our projections Π_N is ultimately not a problem.

We, as usual, take H to be the set of closed subsets of G with the Hausdorff metric (which, extending the abuse of notation, is still again denoted by ρ). We have the usual criterion for tightness (see [FINR]).

For every $N \geq 1$ fix $\varepsilon_N > 0$, $M_N > 0$ and $\delta_N \in [0, 1]^{\mathbb{N}}$ a sequence tending to zero and put $\vartheta = (\varepsilon_N, M_N, \delta_N)_{N \geq 1}$. We denote by $K(\vartheta)$ the set of collections of aged paths in G such that for each collection $C \in K(\vartheta)$ and for each integer $N \geq 1$:

- (i) the number of paths in $\Pi_N(C)$ is less than M_N ;
- (ii) the age of every path in $\Pi_N(C)$ is less than M_N throughout;
- (iii) every path in $\Pi_N(C)$ is contained in $[-N, N] \times [-M_N, M_N]$;

- (iv) every path, $(b, \gamma, a) \in \Pi_N(C)$ has $\omega(2^{-r}, \gamma, [T_N, N]) \leq \delta_N(r)$ for every $r \in \mathbb{N}$, where $T_N = b^{[-N, N]^2}$;
- (v) every path, $(b, \gamma, a) \in \Pi_{N+1}(C)$ has $\gamma(T_N) \in [-N + \varepsilon_N, N - \varepsilon_N]$ and prior to this it did not enter $[-N - \varepsilon_N, N + \varepsilon_N]$;
- (vi) the age process of every path $\gamma \in \Pi_{N+1}(C)$ makes no jump while the age has value in $[2^{-N} - \varepsilon_N, 2^{-N} + \varepsilon_N]$;
- (vii) the age process of every path $\gamma \in \Pi_N(C)$ makes no jumps within time ε_N of each other;
- (viii) the age process of every path $\gamma \in \Pi_{N+1}(C)$ does not have age in interval $(2^{-N} - \varepsilon_N, 2^{-N} + \varepsilon_N)$ at times in $[-N - \varepsilon_N, -N + \varepsilon_N]$ and $[N - \varepsilon_N, N + \varepsilon_N]$.
- (ix) every path, $(b, \gamma, a) \in \Pi_{N+1}(C)$ has $d(\gamma^{[T_N, N]}, \gamma^{[T_N + \eta_1, N + \eta_2]}) \leq 2^{-r}$ for every r with $0 \leq \delta_N(r) < \varepsilon_N/2$ and $0 \leq \eta_1, \eta_2 \leq \delta_N(r)$, where again $T_N = b^{[-N, N]^2}$;

Proposition 9. *The sets $K(\vartheta)$ are compact.*

Proof. Given a sequence of collections C_1, C_2, \dots in K , we prove that we can find a convergent subsequence which converges to an element of K .

If we fix N and consider the paths γ that are in $\Pi_N(C_n)$ for some n , then by conditions (iii) and (iv) this set is compact. Similarly (ii) and (vii) ensure that the age processes will be compact. Since the set of compact sets of trajectories on a bounded domain endowed with the Hausdorff metric is compact, that is $\{\Pi_N(C) | C \in K\}$ is compact for each N , we can take a subsequence, $(C_{n_j^N})$ of (C_n) so that $(\Pi_N(C_{n_j^N}))$ converges. By Cantor diagonal method, we arrive at a subsequence $C_{n_i}, i \geq 1$, such that for every N the sequence $(\Pi_N(C_{n_i}))$ converges to some collection D_N . We must show that from this collection $(D_N)_{N \geq 1}$, we can find an aged path collection C so that C_{n_i} converges to C . It will be clear that any such limit is in $K(\vartheta)$ so the principal task is to produce aged path collection C so that for every N , $\Pi_N(C) = D_N$. The essential step is to show that for each N , $\Pi_N(D_{N+1}) = D_N$. Since $\Pi_N(C_{n_i}) \subset \Pi_N \Pi_{N+1}(C_{n_i})$ It is clear that $D_N \subset \Pi_N(D_{N+1})$. Now we show that $\Pi_N(D_{N+1}) \subset D_N$ by contradiction. Suppose that $\Pi_N(D_{N+1})$ contains a path (b, γ, a) not in D_N . We have by hypothesis that there is a $\delta > 0$ so that (b, γ, a) is distance greater than δ from D_N . We consider a sequence of paths $\gamma_{n_i} \in C_{n_i}$ so that $\Pi_{N+1} \gamma_{n_i}$ converge to (b', γ', a') such that $\Pi_N((b', \gamma', a')) = (b, \gamma, a)$. By condition (v) and (viii) $T_N(\gamma_{n_i})$ must be in time $[-N + \varepsilon_N, N - \varepsilon_N]$ or the age $a(T_N)$ must be greater than $2^{-N} + \varepsilon_N$ and $\gamma_{n_i}(T_N)$ must be in spatial interval $[-N + \varepsilon_N, N - \varepsilon_N]$. From this and (ix), we see that $\Pi_N \gamma_{n_i}$ must converge to (b, γ, a) and the desired contradiction is achieved.

To construct our limit set C (which will clearly be in K), we need to find a collection of aged paths C so that for each N , $\Pi_N C = D_N$. Fix $(\gamma_N, a_N) \in D_N$. By the above paragraph we can find inductively $(\gamma_M, a_M) \in D_M \forall M > N$ so that $\Pi_N((\gamma_M, a_M)) = (\gamma_N, a_N)$. So γ_M and a_M are cadlag functions defined on intervals $[c_M, d_M]$ so that

$$\forall M > M' \geq N, \quad [c_{M'}, d_{M'}] \subset [c_M, d_M] \text{ and } \gamma_M|_{[c_{M'}, d_{M'}]} = \gamma_{M'}, \quad a_M|_{[c_{M'}, d_{M'}]} = a_{M'}.$$

We also have that d_M tends to infinity as M tends to infinity but that (by condition (ii)), $\lim_{M \rightarrow \infty} c_M > -\infty$. We define γ on $(\lim_{M \rightarrow \infty} c_M, \infty)$ by $\gamma(s) = \gamma_M(s)$ for any (and by the consistency all) M with $s \in [c_M, d_M]$. Similarly for a .

We have that (γ, a) has the desired property. We take C to be the totality of paths that can be obtained in this way (i.e. starting from some N and taking a convergent sequence of aged paths. It is clear C is our limit. \square

For $N = 1, 2, \dots$, we consider Γ_N as the random collection of aged paths $\Pi_N(\mathcal{X})$.

Let us fix a space time square $S_N = [-N, N]^2$.

Proposition 10. *Given a path (b, γ, a) that intersects S_N while of age at least ε , let*

$$\beta = \beta(\gamma, \varepsilon) = \inf\{t : (\gamma(t), t) \in S_N \text{ and } (\gamma(t), t) \text{ has age } \geq \varepsilon\}.$$

Given $N \geq 1$ and $1 > \sigma > 0$ then $\exists \eta_1 > 0$ so that

$$P[\exists \gamma \text{ that intersects } S_N \text{ while of age at least } \varepsilon \\ \text{and has variation greater than } \sigma/100 \text{ on } [\beta, \beta + 3\eta_1]] \leq 10^{-6}\sigma^2.$$

Proof. We consider paths γ for which the β as defined above lies in $(i\frac{\varepsilon}{3}, (i+1)\frac{\varepsilon}{3}]$ for fixed i . (There are $\leq \frac{7N}{\varepsilon}$ such i 's for ε small). We fix such an i .

We locally augment notation and write $\beta(\varepsilon)$ as $\beta(\varepsilon, \gamma)$ as we will discuss multiple paths. We are interested in the paths behaviour after the time $\beta(\varepsilon)$. As such it is only necessary to treat a ‘‘good’’ representative. While it may be true that the evolution of the age of path γ immediately before $\beta(\varepsilon, \gamma)$ was very rapid due to several coalescences, for every γ , the behaviour of the path γ on interval $[\beta(\gamma), \infty)$ will equal that of a path γ' on interval $[\beta(\gamma), \infty) = [\beta(\gamma'), \infty)$ for some path whose age at time $i\varepsilon/3$ is at least $\varepsilon/2$. As such to establish the proposition it is enough to treat γ having this property. Henceforth we drop the dependence of β on γ from the notation.

So we are interested in the behaviour on interval $[\beta, \beta + 3\eta_1]$ of paths γ having the property that at time $i\varepsilon/3$ the path has age at least $\varepsilon/2$ and such that in time interval $i\varepsilon/3, (i+1)\varepsilon/3]$ the path γ meets spatial interval $[-N, N]$. We first note that by Proposition 2, the density of the translation invariant collection of walkers of age at least $\varepsilon/2$ at time $i\varepsilon/3$ (or indeed any time) is equal to $K/(\varepsilon/2)^{1/\alpha}$. We now consider (for comparison purposes) the system of stable processes beginning with these walkers evolving independently on time interval $[i\varepsilon/3, (i+1)\varepsilon/3]$ without coalescence. This system at time $(i+1)\varepsilon/3$ will have the same density ($K/(\varepsilon/2)^{1/\alpha}$) and will again be translation invariant. Thus the expected number of points for our comparison system in $[-N, N]$ at time $(i+1)\varepsilon/3$ is exactly $2NK/(\varepsilon/2)^{1/\alpha}$. By the Markov property (applied when a process first enters $[-N, N]$ each process that touches $[-N, N]$ has a probability $\geq C_{N,\varepsilon} > 0$ of being within interval $[-N, N]$ at time $\frac{(i+1)\varepsilon}{3}$ where by symmetry e.g. $C_{N,\varepsilon} > 1/3$ for ε small enough and N large enough. Thus provided ε was fixed sufficiently small, we have that the expectation of the number of comparison processes that touch spatial interval $[-N, N]$ in time interval $[i\varepsilon/3, (i+1)\varepsilon/3]$ is bounded above by

$$6NK/(\varepsilon/2)^{1/\alpha}$$

This bound must then also apply to our original system of coalescing paths by simple stochastic domination.

So the expectation of the total number (i.e. for every relevant i) is thus $\leq \frac{7N}{\varepsilon} \frac{2kN}{(\frac{\varepsilon}{2})^{\frac{1}{\alpha}}}$. By the Markov property (for each stable process) if we choose η_1 so that

$$P(\sup_{s \leq 3\eta_1} |X(s) - X(0)| \geq \sigma/100) \leq \sigma^2 / \left(\frac{7N}{\varepsilon} \frac{2N}{(\frac{\varepsilon}{2})^{\frac{1}{\alpha}}} \cdot 10^7 \right),$$

we have that outside probability $\frac{\sigma^2}{10^7}$, the variation on $[\beta, \beta + 3\eta_1]$ is less than $\sigma/100$ for all these paths. \square

Proposition 11. *Given a path (b, γ, a) that intersects S_N , while of age at least ε , γ , let $\beta = \beta(\gamma, \varepsilon) = \inf\{t : (\gamma(t), t) \in S_N \text{ and } (\gamma(t), t) \text{ has age } \geq \varepsilon\}$. Given $N, \sigma \exists (\delta(r))_{r \geq 1}$ a positive sequence tending to zero as r tends to infinity so that $P[\exists \gamma \text{ that intersects } [-N, N]^2 \text{ while of age at least } \varepsilon \text{ and so that for some } r \geq 1, \omega(2^{-r}, \gamma, [\beta, N]) > \delta(r) \leq \frac{\sigma^2}{10^6}] > 0$.*

Proof. The claim follows from the observations $\omega(2^{-r}, \gamma, [\beta, N]) \rightarrow 0$ as $r \rightarrow \infty$ and the total number of paths is bounded in probability (from the previous proposition). \square

Remark: The results above speak to properties (i) and (iv) in the definition of compact set $K((\varepsilon_N)_{N \geq 1}, (M_N)_{N \geq 1}, (\delta_N)_{N \geq 1})$ while the next proposition addresses (iii). Property (ii) follows from lemma 4

Proposition 12. *Given $\eta_2, \sigma > 0$, there is N' so that the probability that a path (b, γ, a) satisfies*

- (i) γ hits S_N while of age at least 2^{-N} ,
- (ii) there exists $t \in [-N, N]$ so that $a_\gamma(t) \geq \eta_2$ and $\gamma(t) \in [-N', N']^c$,

is bounded by σ .

Proof. Fix a positive integer k and let A_k be the event that for some i , there is a path γ that intersects I_k in time $\left[\frac{i\eta_2}{3}, (i+1)\frac{\eta_2}{3}\right]$ so that

- (i) $\left[\frac{i\eta_2}{3}, (i+1)\frac{\eta_2}{3}\right] \cap [-N, N] \neq \emptyset$;
- (ii) the path has age $\geq \frac{\eta_2}{2}$ at time $\frac{i\eta_2}{3}$;
- (iii) $I_k = [2^k N^\beta, 2^{k+1} N^\beta] \cup [-2^{k+1} N^\beta, -2^k N^\beta]$ for $\beta \geq \frac{2}{\alpha}$.

Then we have since the number of such i is bounded above by $\frac{7N}{\eta_2}$ and the expected number of such paths is dominated by $\frac{K2^k N^\beta}{\eta_2^{\frac{1}{\alpha}}}$ then the expectation of this number is bounded above by

$$\frac{7N}{\eta_2} \frac{K2^k N^\beta}{\eta_2^{\frac{1}{\alpha}}} q_N.$$

where q_N is the probability that a stable process starting at some point in interval $[-N, N]$ visits I_k before time $2N$ which is of order $N/N^{\beta(\alpha+1)}2^{k(\alpha+1)}$. Thus for some $K' > 0$

$$P(A_k) \leq \frac{7KK'}{\eta_2^{1+\frac{1}{\alpha}}} N^{2-\beta\alpha} 2^{-k\alpha}.$$

Since $\beta \geq \frac{2}{\alpha}$, summing this over $k \geq k_0$ gives $P(\cup_{k \geq k_0} A_k) < \sigma$ for k_0 sufficiently large. Since $\cup_{k \geq k_0} A_k$ contains the event in the statement we are done. \square

We use Proposition 12 above to show:

Lemma 13. *For every $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\eta > 0$ so that*

$P(\exists (b, \gamma, a)$ so that $\Pi_N \gamma \neq \emptyset$ and so that $a(\cdot)$ jumps twice in an interval of length η) is less than ε .

Remark: Lemma 13 above addresses (vii) of the definition of the compact sets $K(\vartheta)$.

Proof. We fix N and ε . First pick N' according to Proposition 12 applied to N with $\eta_2 = 2^{-N}/10$ and $\sigma = \varepsilon/100$. Then pick N'' in this way with N' substituted for N . Pick M so that the probability that the number of paths in $\Phi_{N+2}(C)$ to touch $(-N'', N'') \times (-N, N)$ is greater than M is less than $\varepsilon/100$. Let the complements of these "expected events" be denoted by B_1, B_2, B_3 :

B_1 is the event that there exists a path which touches spatial interval $[-N, N]$ in time interval $[-N - 1, N + 1]$ while having been outside spatial interval $[-N', N']$ while of age greater than $\eta_2 = 2^{-N}/10$,

B_2 is the event that there exists a path which touches spatial interval $[-N', N']$ in time interval $[-N - 1, N + 1]$ while having been outside spatial interval $[-N'', N'']$ while of age greater than $\eta_2 = 2^{-N}/10$,

B_3 is the event that the number of paths to touch spatial interval $[-N'', N'']$ during temporal interval $[-N - 1, N + 1]$ while having age greater than 2^{-N-2} exceeds M .

We divide up the event in question into the union of events $A(i, N)$ is the event that a path of age $\geq 2^{-N}$ meets two paths also of age $\geq 2^{-N}$ in time interval $[i\eta, (i + 2)\eta]$ (which intersects $[-N, N]$) and all three paths were in $(-N'', N'')$ at time $i\eta - 2^{-N-1}$. It is easy to see that the probability of $\cup_{i\eta \in (-N-1, N)} A(i, N)$ occurring but not one of the B_i is less than $Const(N/\eta)2^{2N/\alpha}M^3\eta^{2/\alpha}$: we simply note that for a fixed such interval, for the event to occur (and none of the B_i) at time $i\eta - 2^{-N}/4$ all three of the processes must be among the at most M processes of age at least $2^{-N}/2$ in spatial interval $[-N'', N'']$. We have at most M^3 choices for the three processes. Uniformly over the positions of the three processes a time $i\eta - 2^{-N}/4$, the probability that the first two meet and then the third in the given time interval is bounded by $\eta^{2/\alpha}2^{2N/\alpha}$.

Given that our value M has been fixed (and that $\alpha < 2$) this upper bound will be less than $\varepsilon/100$ if η is chosen small enough. Thus with these choices of N', n'', M and η the probability of the event occurring is less than $\varepsilon/100 + \varepsilon/100 + \varepsilon + 100 + \varepsilon/100 < \varepsilon$ and the result follows. □

In a similar way we can show the following which is relevant to (vi) of the definition of $K(\vartheta)$.

Proposition 14. *For each $\sigma, \varepsilon > 0$, $N < \infty$, $\exists \eta > 0$ so that*

$$P[\exists \text{ a path that comes within } \eta \text{ of an older path while having age in } (\varepsilon - \eta, \varepsilon + \eta) \text{ while in } [-N, N]^2] < \sigma^2/10^6$$

Again by Proposition 12 we can restrict attention to paths within N' of the origin. Again we bound the number of such paths that touch this spatial interval during time interval $(-N, N)$ while of age at least 2^{-N-1} . The argument is now as with Lemma 13

Proposition 15. *For each $\sigma, \varepsilon > 0$, $N < \infty$, $\exists \varepsilon_N > 0$ so that for event $A(N, \varepsilon_N) \equiv \{\exists \gamma \text{ having age in interval } (2^{-N} - \varepsilon_N, 2^{-N} + \varepsilon_N) \text{ at times in } (-N - \varepsilon_N, -N + \varepsilon_N) \text{ and } (N - \varepsilon_N, N + \varepsilon_N) \text{ while in interval spatial } [-N, N]\}$ satisfies*

$$P[A(N, \varepsilon_N)] < \sigma^2/10^6.$$

To see this for the time interval $(N - \varepsilon_N, N + \varepsilon_N)$, we first choose N' so that (using Proposition 12), outside a set of probability $\sigma^2/10^7$. any path that meets $(-N - 1, N + 1)$ in time interval $(-N - 1, N + 1)$ must be within N' of the origin while having age at least $2^{-N}/3$. Outside this small probability event, the claimed event lies in the existence of a path such that at time $N - 2^{-N-1}$ lies in $(-N', N')$ and has age in interval of length $4\varepsilon_N$ around 2^{-N-1} . Given Corollary 3, we obtain the result.

We can equally address property (v) in our definition of compact K :

Lemma 16. *For each N and each $\delta > 0$, $\exists \varepsilon_N > 0$ so that the probability that there exists a path γ which first hits $[N - \varepsilon_N, N + \varepsilon_N]$ at a time before or equal to its first time of hitting $[-N, N]^2$ while of age at least 2^{-N} is less than δ .*

Proof. We denote the “bad” event whose probability we wish to bound by B_N . It follows from the self similarity properties of the stable process that for a stable process $\{X(s)\}_{s \geq 0}$ starting at 1 with $\tau = \inf\{s : X(s) \leq 0\}$, we have $X(\tau) < 0$. By quasi left continuity we get that

$$c(\varepsilon) \equiv P^1(\{X(s) \ 0 \leq s \leq \tau\} \cap [-\varepsilon, \varepsilon] \neq \emptyset) \Rightarrow 0$$

as ε tends to zero. So by scaling we have for τ now equal to $\inf\{s : X(s)X(0) \leq 0\}$

$$\sup_{|x| \geq a} P^x(\{X(s) \ 0 \leq s \leq \tau\} \cap [-\varepsilon, \varepsilon] \neq \emptyset) = c(\varepsilon/a).$$

We divide up B_N into four parts:

$B_N(1)$: a path enters $[-N, N]^2$ while of age at least 2^{-N} which had been outside spatial interval $[-N', N']$ while of age in $[2^{-N}/3, 2^{-N}]$.

$B_N(2)$: the number of paths inside $[-N', N'] \times [-N, N]$ of age at least $2^{-N}/3$ is greater than N''

$B_N(3)$: there exists a path that achieves age 2^{-N} while spatially in $[N - \varepsilon'_N, N + \varepsilon'_N]$.

$B_N(4)$: B_N occurs through a path that hits $[-N, N]^2$ with age at least 2^{-N} which achieved age 2^{-N} while outside $[N - \varepsilon'_N, N + \varepsilon'_N]$

N' , N'' and ε'_N will be specified as the proof progresses.

By Proposition 12 if N' is fixed high enough, then $P(B_N(1)) < \delta/4$. Similarly we have that for N'' sufficiently large $P(B_N(2) \setminus B_N(1)) < \delta/4$. By applying the Markov property at age time $2^{-N}/3$ we easily see that $P(B_N(3) \setminus (B_N(1) \cup B_N(2))) < CN''\varepsilon'_N 2^{N/\alpha} < \delta/4$ if ε'_N is fixed small enough. Finally

$$P(B_N(4) \setminus (B_N(1) \cup B_N(2) \cup B_N(3))) < CN''c\left(\frac{\varepsilon_N}{\varepsilon'_N}\right)$$

which is less than $\delta/4$ if ε_N is chosen small enough. \square

Propositions 11, 14, 15, Lemma 13 and Corollary 8 as well as the proof of Proposition 10 yield.

Proposition 17. For each $\sigma, \exists \vartheta = (\varepsilon_N)_{N \geq 1}, (M_N)_{N \geq 1}, (\delta_N)_{N \geq 1}$ so that

$$P[C \notin K(\vartheta)] < \sigma^2/10^6$$

Remark: This shows that our measure on aged paths is tight given Proposition 9

A θ process is simply a stable process beginning at a some time point in $\theta\mathbb{Z}^2$. The $(\theta i, \theta j)$ -process is simply the stable process beginning at position θi at time θj (so that θ processes are simply the collection over $(i, j) \in \mathbb{Z}^2$ of $(\theta i, \theta j)$ -processes.

Proposition 18. Given $N, N', \eta, \varepsilon \exists \theta > 0$ so that $P[\exists$ a path so that $(\gamma(s), s) \in [-N', N'] \times [-N', N]$ while having age $\geq \eta$ but is not ‘coalesced’ with a $(\theta i, \theta j)$ process by age $2\eta] < \varepsilon$ for $(\theta i, \theta j) \in [-N', N'] \times [-N', N]$

(Note: we can define the “age” for a θ process $X^{i\theta, j\theta}$ (considered among other θ -processes) at time $s > j\theta$ as the largest value $s - j'\theta$ over j' so that for some i' , $X_s^{i\theta, j\theta} = X_s^{i'\theta, j'\theta}$.

Proof. We simply consider processes that at a time $\frac{i\eta}{3}$ in $\left(\frac{\eta}{3}, \frac{2\eta}{3}\right)$ have life for some i with $\frac{i\eta}{3} \in [-N, N]$. Fix (provisionally) an i .

For each one enumerated in some arbitrary manner we can apply the Markov property at the time τ the age becomes η in the following at times $\tau, \tau + \theta, \tau + 2\theta, \dots$ there is a chance bounded

away from 0 that the stable process will coalesce with the θ -process nearest it, so the probability that the coalescence is not effected by time age 2η is $\leq (1-C)^{\lfloor \frac{2}{\theta} \rfloor}$, so the expected number of non coalesced processes is bounded (as usual) by $\frac{7N}{\eta} \times \frac{KN'}{(\eta')^{1/\alpha}} \times (1-C)^{\lfloor \frac{2}{\theta} \rfloor} < \sigma$ for θ small. C here is (by scaling) the infimum over $|x| \leq 1$ of the probability that two independent stable processes beginning at time 0 at x and 0 hit before time 1. \square

This begets

Corollary 19. *Given N and $\sigma > 0$, there exists N' and $\theta > 0$ so that if L is the collection of coalescing stable processes in $\theta\mathbb{Z}^2 \cap [-N', N'] \times [-N', N]$, then outside probability σ for every path γ with $(\gamma(s), s) \in [-N, N]^2$ for some $s \in [-N, N]$ with $a(s) \geq 2^{-N}$, we have $\gamma' \in L$ with distance*

$$\rho(\Psi_N \circ \Phi_{2^{-N}}\gamma, \Psi_N \circ \Phi_{2^{-N}}\gamma') < \sigma/10$$

where by abuse of notation $\Psi_N \circ \Phi_{2^{-N}}\gamma'$ is the path with the operator $\Phi_{2^{-N}}$ applied for the age of γ' among θ processes.

Proof. Given N and σ let us apply Proposition 10 with $\varepsilon = 2^{-N}$ to obtain η_1 satisfying the desired condition. We can also, arguing as in Lemma 13 suppose that η_1 is sufficiently small that no path hitting $[-N, N]^2$ while of age greater than 2^{-N} has a jump in $[a(2^{-N}) - \eta_1, a(2^{-N}) + \eta_1]$ outside this probability. Now given this η_1 (which we can take to be small compared to 2^{-N} , let η_2 be less than $\sigma\eta_1/100$. We apply Proposition 12 with $\eta = \eta_2$ and σ equal to our fixed $\sigma^2/10^6$. This yields our desired N' (We here also suppose that outside this probability no path hitting $[-N, N]^2$ has age greater than N'). Applying Proposition 18, with N, N', η and $\varepsilon = \sigma^2/10^6$ we have our θ and outside of probability $2\sigma^2/10^6$, every path γ as above has coalesced with a path in L before it has age 2η . The result now follows from Proposition 10. \square

Propositions 10, 12 and 18 and Corollary 19 yield

Proposition 20. *$\forall \sigma > 0$ there exists N, N', θ so that outside probability σ the ρ distance between the paths resulting from the stable web and the paths resulting from the $\theta\mathbb{Z}^2 \cap [-N', N'] \times [-N', N+1]$ beginning after age 2^{-N} (with age as mentioned after Proposition 18) when the processes touch $[-N, N]$ is less than σ .*

We denote the system of θ -processes by \mathcal{X}^θ and the system of θ -processes of age at least δ by $\mathcal{X}_\delta^\theta$. We claim that \mathcal{X}^θ plays the role of a skeleton for the stable web in an analogous way to the definition of a skeleton for the Brownian Web, see [FINR].

4. Convergence in Distribution

Consider a random walk $(W_n)_{n \geq 1}$ such that $W_n = \sum_{i=1}^n Z_n$ with $(Z_n)_{n=1}^\infty$ iid random variables whose distribution is in the domain of attraction of a stable symmetric $\alpha \in (0, 2)$ r. v. $X = X_1^{0,0}$ ($X^{0,0}$ defined as in Section 2). Let $p(x) = P(Z_1 = x)$, $x \in \mathbb{Z}$ be its transition probability function. Since the best convergence result is not our focus we assume that $p(\cdot)$ is symmetric and satisfies

$$x^{1+\alpha}p(x) \rightarrow C \in (0, \infty), \quad \text{as } |x| \rightarrow \infty,$$

where the C is chosen to be compatible with X . Thus we have that

$$\left(\frac{W_{\lfloor nt \rfloor}}{n^{\frac{1}{\alpha}}} \right)_{t \geq 0}$$

converges in distribution under the Skorohod topology to $(X_t^{0,0})_{t \geq 0}$. We have the Gnedenko local CLT, see [?],

$$\sup_{|x_0| \leq K\eta^{\frac{1}{\alpha}}} \left| n^{\frac{1}{\alpha}} P(W_n = x_0) - P_X \left(\frac{x_0}{n^{\frac{1}{\alpha}}} \right) \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This is enough to show convergence of the Green functions:

$$n^{(\frac{1}{\alpha}-1)} \sum_{j=1}^{\infty} P(W_j = [un^{\frac{1}{\alpha}}]) e^{-\beta \frac{j}{n}} \rightarrow G_{\beta}(u), \quad \forall \beta > 0,$$

uniformly on compact intervals for u .

From this and standard optional stopping, we have

Lemma 21. *For every $\beta > 0$, if W^1, W^2 are two iid random walks with increments distributed as $p(\cdot)$ and starting at $W_0^1 = 0$ and $W_0^2 = [n^{\frac{1}{\alpha}}u]$ and $T_n = \frac{1}{n} \inf\{j : W_j^1 = W_j^2\}$ then $\lim_{n \rightarrow \infty} Ee^{-\beta T_n} = Ee^{-\beta T}$ where $T = \inf\{t : X_t^1 = X_t^2\}$ for X^1, X^2 iid stable processes distributed as $X^{0,0}$ starting at $X_0^1 = 0$ and $X_u^2 = u$. Therefore $T_n \Rightarrow T$.*

From this we obtain

Lemma 22. *For N, N^1, ε and θ fixed positive and finite, the system of coalescing stable processes starting from points in $\theta\mathbb{Z}^2 \cap [-N^1, N^1] \times [N, N]$ is the limit of the system of coalescing random walks starting on $[-N^1 n^{\frac{1}{\alpha}}, N^1 n^{\frac{1}{\alpha}}] \times [-Nn, Nn]$ starting from points in $(\theta n^{\frac{1}{\alpha}}\mathbb{Z} \times \theta n\mathbb{Z})$ and appropriately rescaled.*

Recall the definitions of Section 2 and Proposition 2. The system \bar{X} of α -stable processes starting from full occupancy at time 0 is scale invariant and particular the density scales as $k/t^{\frac{1}{\alpha}}$ for some constant k not depending on t . We now note that the density for coalescing random walks scales (when suitably renormalized) in the same way.

Proposition 23. *For coalescing random walks on $\mathbb{Z} \times \mathbb{R}_+$ beginning with full occupancy the density at time $n \approx k/n^{\frac{1}{\alpha}}$ where k is the constant for the continuous time coalescing processes obtained in Proposition 2.*

Proof. For the system of α -stable coalescing processes the density at time t is the (increasing) limit as $\theta \downarrow 0$ of the processes beginning at $\theta\mathbb{Z}$. Thus for every $\varepsilon > 0$, there exists $\theta > 0$ so that the density of coalescing θ -processes at given times t_1, t_2 are greater than $(k - \varepsilon)/t_1^{\frac{1}{\alpha}}$ and $(k - \varepsilon)/t_2^{\frac{1}{\alpha}}$.

We first take $t_1 = 1$. Now for θ as above, we consider the coalescing random walks beginning at $\theta n^{\frac{1}{\alpha}}\mathbb{Z}$. By the invariance principle and following discussion we get that for n large the density of these coalescing random walks is at least $(k - 2\varepsilon)/n^{\frac{1}{\alpha}}$ for n large. Hence by monotonicity it is at least $(k - 2\varepsilon)/n^{\frac{1}{\alpha}}$ for the full process of coalescing random walks (i.e. starting from full occupancy).

On the other hand we can via Bramson and Griffeath arguments [BG] show that there exists $m < \infty$ so that $\forall n$ the density of coalescing random walks at time T is bounded above by $m/n^{\frac{1}{\alpha}}$. In particular at $T = rn, r$ small, the density is bounded above by $m/(r^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}})$. We now couple this to a coalescing system of random walks starting with occupancy at $\theta n^{\frac{1}{\alpha}}\mathbb{Z}$ at time $T = rn$ and we take $t_2 = (1 - r)$ then the density of the full process of coalescing random walks at time $n = rn + (1 - r)n$ is equal to the density of the $\theta n^{\frac{1}{\alpha}}\mathbb{Z}$ random walks

at time $(1-r)n$ plus the density of walks of the full process that have not coupled with a θ random walk by time n . But (if θ is sufficiently small) this latter density will be smaller than ε while the former density (by invariance) will be less than $(k+\varepsilon)/((1-l)^{\frac{1}{\alpha}}n^{\frac{1}{\alpha}})$. Thus the density of full random walks at time n will be bounded above by

$$\frac{k+\varepsilon}{(1-l)^{\frac{1}{\alpha}}n^{\frac{1}{\alpha}}} + \varepsilon.$$

We now let $r \downarrow 0$ and then $\varepsilon \downarrow 0$ to obtain our result. \square

From now on we work with systems of continuous time coalescing random walks starting at points on $n^{-\frac{1}{\alpha}}\mathbb{Z} \times n^{-1}\mathbb{Z}$, making jumps from $in^{-1/\alpha}$ to $jn^{-1/\alpha}$ at rate $np(j-i)$. The age of such a random walk is defined to be continuously increasing at rate 1 on intervals of no coalescence and when a coalescence occurs the age jumps to the age of the older path. We call it an aged random walk. We denote the collection of aged random walks (suitably renormalized) as \mathcal{W}_n for $n \geq 1$ (so n is the scaling parameter). Given Proposition 23 we can easily prove the following analogue of Proposition 4:

Proposition 24. *Given an aged random walk path $(\gamma, a) \in \mathcal{W}_n$ that intersects S_N , while of age at least ε , let*

$$\beta = \beta(\gamma, \varepsilon) = \inf\{t : (\gamma(t), t) \in S_N \text{ and } a(t) \geq \varepsilon\}.$$

Given N and σ there exists η_1 so that

$$P[\exists (\gamma, a) \in \mathcal{W}_n \text{ that intersects } S_N \text{ while of age at least } \varepsilon \text{ and has variation}$$

$$\text{greater than } \sigma \text{ on } [\beta, \beta + 3\eta_1]] \leq \frac{\sigma^2}{10^6},$$

for every n .

We similarly have analogues of Propositions 10, 11, 14, 25, 12, 18 and 20 as well as Lemmas 13 and 16. This yields:

Proposition 25. *For each $\sigma > 0$, there exists $\vartheta = ((\varepsilon_N)_{N \geq 1}, (M_N)_{N \geq 1}, (\delta_N)_{N \geq 1})$ so that for each $\delta > 0$, there exists $n_0 = n_0(\delta) < \infty$ so that for $n \geq n_0$*

$$P[\mathcal{W}_n \notin K^\delta(\vartheta)] < \sigma^2/10^6.$$

In particular $(\mathcal{W}_n)_{n \geq 1}$ is a tight family of random elements of H .

Remark: We need to consider K^δ rather than simply K since the convergence of renormalized random walks to continuous time stable processes ensures the desired convergence for paths $\Pi_N \gamma$ for n large enough (depending on N) a

The aim from this point is to prove weak convergence of \mathcal{W}_n to \mathcal{X} . Our argument uses the approximation of \mathcal{X} by θ -processes and we need an analogous approximation for the system of aged random walks. So we consider the system of θ -random walks associated to the scaling parameter n as the collection of rescaled coalescing random walks starting at $\theta\mathbb{Z}^2$ that evolves as before, i.e. making jumps from $in^{-1/\alpha}$ to $jn^{-1/\alpha}$ at rate $np(j-i)$.

Given a collection of θ -random walks, we can, just as in the original process, speak of ages of paths: the age of a path $X^{\theta i, \theta j}$ at time $s > j\theta$ is simply

$$s - \inf\{\theta l : \exists k : X_s^{\theta i, \theta j} = X_s^{\theta k, \theta l}\}.$$

We then denote by \mathcal{W}_n^θ the system of aged coalescing θ -random walks with scaling parameter n and given N and N' we write $\mathcal{W}_n^{\theta, N'}$ for the system of coalescing θ -random walks with

scaling parameter n beginning at points $\theta\mathbb{Z}^2 \cap [-N-1, N] \times [-N', N']$. (Usually N is given and so is dropped from the notation.)

As in Proposition 25, we can show that $(\mathcal{W}_n^\theta)_{n \geq 1}$ is a tight family of random elements of H . Moreover, we can prove as in Corollary 19 the following result:

Proposition 26. *Given N , N' and θ , the coalescing renormalized systems $\mathcal{W}_n^{\theta, N'}$ converge in distribution to $\mathcal{X}^{\theta, N'}$ as n tends to infinity. Furthermore $\Psi_N \circ \Phi_{2^{-N}}(\mathcal{W}_n^{\theta, N'})$ converges in distribution to $\Psi_N \circ \Phi_{2^{-N}}(\mathcal{X}^{\theta, N'})$.*

We are now ready to establish weak convergence of \mathcal{W}_n to \mathcal{X} .

To establish weak convergence it is sufficient to show that for a bounded and continuous F on our space

$$E[F(\mathcal{W}_n)] \rightarrow E[F(\mathcal{X})]$$

as n tends to infinity.

We fix $\varepsilon > 0$ and below for a set $K \in H$ we denote $K^\eta = \{\psi : \rho(\psi, K) \leq \eta\}$.

Now we fix a bounded continuous function F on the set of aged path collections. Given $\varepsilon > 0$, we fix a compact set K of collections of paths as in Proposition 9 so that the probability that $\mathcal{X} \in K$ is at least $1 - \varepsilon/3$. By the compactness of K we have that there exists $\eta' > 0$ so that

$$\forall \psi \in K, \quad \sup_{\psi': \rho(\psi, \psi') < 100\eta'} |F(\psi) - F(\psi')| < \varepsilon/100$$

which immediately implies that

$$\forall \psi \in K^{\eta/50}, \quad \sup_{\psi': \rho(\psi, \psi') < 50\eta'} |F(\psi) - F(\psi')| < \varepsilon/50.$$

By Proposition 26 for any $\theta > 0$ (and N, N'),

$$|E[F(\mathcal{W}_n^\theta)] - E[F(\mathcal{X}^\theta)]| \rightarrow 0.$$

We choose θ , N and N' according to Proposition 20 so that the distance between \mathcal{X} and \mathcal{X}^θ is less than η' outside probability $\varepsilon/10$. We can take this θ , N and N' so that we have equally for each n , the distance between \mathcal{W}_n and \mathcal{W}_n^θ is less than η' outside this probability. We now have

$$\begin{aligned} |E[F(\mathcal{X})] - E[F(\mathcal{X}^\theta)]| &\leq E[|F(\mathcal{X}) - F(\mathcal{X}^\theta)|] \\ &\leq E[|F(\mathcal{X}) - F(\mathcal{X}^\theta)| I_{\mathcal{X} \notin K}] \\ &\quad + E[|F(\mathcal{X}) - F(\mathcal{X}^\theta)| I_{\rho(\mathcal{X}, \mathcal{X}^\theta) < \eta} I_{\mathcal{X} \in K}] \\ &\quad + E[|F(\mathcal{X}) - F(\mathcal{X}^\theta)| I_{\rho(\mathcal{X}, \mathcal{X}^\theta) \geq \eta}]. \end{aligned}$$

This latter sum is bounded by $2\varepsilon\|F\|_\infty + \varepsilon + 2\varepsilon\|F\|_\infty \leq 5\varepsilon(1 \vee \|F\|_\infty)$ by Proposition 20.

We also have that for any $\eta > 0$, $P(\mathcal{W}_n \in K^\eta) > 1 - \varepsilon/2$ for n large and so we can argue as above that for universal C

$$|E[F(\mathcal{W}_n)] - E[F(\mathcal{W}_n^\theta)]| \leq C\varepsilon(1 \vee \|F\|_\infty)$$

for n large which gives that $|E[F(\mathcal{W}_n)] - E[F(\mathcal{X})]| \leq C'\varepsilon(1 \vee \|F\|_\infty)$ for n large.

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