

DYNAMICAL PROPERTIES OF RANDOM WALKS

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ABSTRACT. In this paper, we study dynamical properties as hypercyclicity, supercyclicity, frequent hypercyclicity and chaoticity for transition operators associated to countable irreducible Markov chains. As particular cases, we consider simple random walks on \mathbb{Z} and \mathbb{Z}_+ .

1. INTRODUCTION

Let X be a Banach separable space on \mathbb{C} and $T : X \rightarrow X$ be a linear operator on X . The study of the linear dynamical system (X, T) became very active after 1982. Since then related works have built connections between dynamical systems, ergodic theory and functional analysis. We refer the reader to the books [2, 7] and to the more recent papers [3, 4, 5, 11, 12], where many additional references can be found.

The objective of this paper is to study some central properties of linear dynamical systems as hypercyclicity, supercyclicity, frequent hypercyclicity, and chaoticity among others, for Markov chain transition operators associated to countable irreducible Markov chains. In particular, we will consider to nearest-neighbor simple random walks.

We say that (X, T) is *hypercyclic*, or topologically transitive, if it has a dense orbit in X . This notion is equivalent that for all non empty open subsets U and V of X , there exists an integer $n \geq 0$ such that $T^n(U) \cap V$ is not empty. If moreover for every non-empty open set $V \subset X$, the set $N(x, V) = \{k \in \mathbb{N}, T^k(x) \in V\}$ has positive lower density, i.e $\liminf_{n \rightarrow \infty} \frac{1}{n} \text{card}(N(x, V) \cap [1, n]) > 0$, then we call (X, T) *frequently hypercyclic*. On the other hand, (X, T) is said to be *supercyclic* if there exists $x \in X$ such that the projective orbit of x is dense in the sphere $S^1 = \{z \in X, \|z\| = 1\}$, that is the set $\{\lambda T^n(x), n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in X . We call (X, T) *Devaney chaotic* if it is hypercyclic, has a dense set of periodic points and has a sensitive dependence on the initial conditions.

The study of those four properties is a central problem in area of linear dynamical systems (see for instance [2] and [7]). Notice that the above properties can be studied in the context of more general topological space X called Frechet spaces (the topology is induced by a sequence of semi-norms).

There are many examples of hypercyclic linear operators (see [2]) as the derivative operator on the Frechet space $H(\mathbb{C})$ of holomorphic maps on \mathbb{C} endowed

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with the topology of uniform convergence on compact sets, translation operator on $H(\mathbb{C})$, classes of weighted shift operators acting on $X \in \{c_0, l^p, p \geq 1\}$.

However, the set of hypercyclic linear operators is small. In fact it is proved that this set is nowhere dense in the set of continuous linear operators with respect to the norm topology (see [2]). An example of non hypercyclic operator is the shift operator S acting on $X \in \{c_0, l^q, q \geq 1\}$. This comes from the fact that the norm of S is less or equal to 1. However, the shift operator is supercyclic and moreover for any $\lambda > 1$, λS is frequently hypercyclic and chaotic (see [2]).

Here we are interested in operators associated to stochastic infinite matrices acting on a separable Banach space $X \in \{c_0, c, l^q, q \geq 1\}$. In particular, we prove that if A is a transition operator on an irreducible Markov chain with countable state space acting on c , then A is not supercyclic. The result remains valid if we replace c by c_0 or l^q , $q \geq 1$ in the positive recurrent case. A natural question is: what happens when the Markov chain is null recurrent or transient if $X \in \{c_0, l^q, q \geq 1\}$? In order to study the last question, we consider transition operators W_p (resp. \overline{W}_p) of nearest-neighbor simple asymmetric random walks on \mathbb{Z}_+ (resp. on \mathbb{Z}) with jump probability $p \in (0, 1)$.

For the simple asymmetric random walk on \mathbb{Z}^+ defined in $X \in \{c_0, c, l^q, q > 1\}$, we prove that if the random walk is transient ($p > 1/2$), then W_p is supercyclic and moreover for all $|\lambda| > \frac{1}{2p-1}$, λW_p is frequently hypercyclic and chaotic. If the random walk is null recurrent ($p = 1/2$) and $X = l^1$, then W_p is not supercyclic.

For the simple asymmetric random walk on \mathbb{Z} , we prove that if $p \neq 1/2$ (transient case), $\lambda \overline{W}_p$ is not hypercyclic for all $|\lambda| > \frac{1}{|1-2p|}$.

We also consider transition operators spatially inhomogeneous simple random walks on \mathbb{Z}_+ , that is operators $G_{\bar{p}} := G$ associated to a sequence of probabilities $\bar{p} = (p_n)_{n \geq 0}$ and defined by $G_{0,0} = 1 - p_0$, $G_{0,1} = p_0$ and for all $i \geq 1$, $G_{i,j} = 0$ if $j \notin \{i-1, i+1\}$, $G_{i,i-1} = 1 - p_i$ and $G_{i,i+1} = p_i$.

In particular, we prove the following result: Consider the sequence

$$w_n = \frac{(1-p_1)(1-p_3)\dots(1-p_{n-1})}{p_1 p_3 \dots p_{n-1}} \text{ for } n \text{ even,}$$

and

$$w_n = \frac{(1-p_0)(1-p_2)\dots(1-p_{n-3})(1-p_{n-1})}{p_0 p_2 \dots p_{n-3} p_{n-1}} \text{ for } n \text{ odd.}$$

The following results hold:

1. If $X = c_0$ and $\lim w_n = 0$ or $X = l^q$, $q \geq 1$ and $\sum_{n=1}^{+\infty} w_n^q < +\infty$, then G is supercyclic on X .

2. Let $X \in \{c_0, l^q, q \geq 1\}$ and assume that there exists $\alpha > 0$ such that $p_n \geq \frac{1}{2} + \alpha$ for all $n \geq n_0$, then there exists $\delta > 1$ such that λG is frequently hypercyclic and Devaney chaotic for all $|\lambda| > \delta$.

The last two results can be extended for the spatially inhomogeneous simple random walks on \mathbb{Z} .

As a consequence of our dynamical study of random walks, we deduce that if the Markov chain is null recurrent, it cannot be supercyclic on l^1 (see Proposition 4.6) or supercyclic on c_0 (see remark 4.4). We also deduce that, when

the Markov chain is transient, it can have nice dynamical properties as supercyclicity, frequently hypercyclicity and chaoticity on X in $\{c_0, l^p, p \geq 1\}$ (see Theorems 4.1 and 4.8). We wonder if it is possible to construct transient Markov chains on \mathbb{Z}_+ or \mathbb{Z} that are not supercyclic.

The paper is organized as follows: In section 2, we give some definitions and classical results. Section 3 describes the study of dynamical properties of Markov chain operators. In section 4, we consider operators associated to the simple asymmetric and also the spatially inhomogeneous simple random walks on \mathbb{Z}_+ and \mathbb{Z} .

2. DEFINITIONS AND CLASSICAL RESULTS

To fix the notation we introduce here the proper definitions of the spaces mentioned above: Let $w = (w_n)_{n \geq 0}$ be a sequence of complex numbers. We put

$$\|w\|_\infty = \sup_{n \geq 0} |w_n| < \infty, \quad \|w\|_q = \left(\sum_{n \geq 0} |w_n|^q \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty,$$

and

$$\begin{aligned} l^\infty &= l^\infty(\mathbb{Z}_+) = \{w \in \mathbb{C}^{\mathbb{Z}_+} : \|w\|_\infty < \infty\}, \\ l^q &= l^q(\mathbb{Z}_+) = \{w \in \mathbb{C}^{\mathbb{Z}_+} : \|w\|_q < \infty\}, \\ c &= c(\mathbb{Z}_+) = \{w \in l^\infty : w \text{ is convergent}\}, \\ c_0 &= c_0(\mathbb{Z}_+) = \{w \in c : \lim_{n \rightarrow \infty} w_n = 0\}. \end{aligned}$$

Now recall the definitions from the introduction. Other definitions related to linear dynamics will be needed.

Definition 2.1. *Let $f : Y \rightarrow Y$ be a continuous map acting on some metric space (Y, d) . We say that f is Devaney chaotic if*

- (1) *f is hypercyclic;*
- (2) *f has a dense set of periodic points;*
- (3) *f has a sensitive dependence on initial conditions: there exists $\delta > 0$ such that, for any x in Y and every neighborhood U of x , one can find $y \in U$ and an integer $n > 0$ such that $d(f^n(x), f^n(y)) \geq \delta$.*

Let X be a Banach separable space on \mathbb{C} and $T : X \rightarrow X$ be a linear bounded operator on X .

Definition 2.2. (see [2]). *We say that T satisfies the hypercyclicity (resp. supercyclicity) criterion if there exists an increasing sequence of nonnegative integers $(n_k)_{k \geq 0}$, two dense subspaces of X , D_1 and D_2 and a sequence of maps $S_{n_k} : D_2 \rightarrow X$ such that*

- (1) $\lim_k T^{n_k}(x) = \lim_k S_{n_k}(y) = 0$ (resp. $\lim_k \|T^{n_k}(x)\| \|S_{n_k}(y)\| = 0$), $\forall x \in D_1, y \in D_2$.
- (2) $\lim_k T^{n_k} \circ S_{n_k}(x) = x, \forall x \in D_2$.

Theorem 2.3. *The following properties are true*

- (1) *If T satisfies the hypercyclicity criterion, then T is hypercyclic.*

- (2) T satisfies the hypercyclicity criterion if and only if T is topologically weakly mixing, i.e $T \times T$ is topologically mixing.
- (3) If T satisfies the supercyclicity criterion, then T is supercyclic.

There is an efficient criterion that guarantees that T is Devaney chaotic and frequently hypercyclic (see [2]).

Theorem 2.4. *Assume that there exist a dense set $D \subset X$ and a map $S : D \rightarrow D$ such that*

- (1) *For any $x \in D$, the series $\sum_{n=0}^{+\infty} T^n(x)$ and $\sum_{n=0}^{+\infty} S^n(x)$ are unconditionally convergent (all subseries of both series are convergent).*
- (2) *For every $x \in D$, $T \circ S(x) = x$,*

then T is chaotic and frequently hypercyclic.

Concerning the dynamical properties of a linear dynamical system (X, T) , the spectrum of T plays an important role. We denote by $\sigma(X, T)$, $\sigma_{pt}(X, T)$, $\sigma_r(X, T)$ and $\sigma_c(X, T)$ respectively the spectrum, point spectrum, residual spectrum and continuous spectrum of T . Recall that λ belongs to $\sigma(X, T)$ (resp. $\sigma_{pt}(X, T)$) if $(S - \lambda I)$ is not bijective (resp. not one to one). If $(S - \lambda I)$ is one to one and not onto, then $\lambda \in \sigma_r(X, T)$ if $(S - \lambda I)(X)$ is not dense in X , otherwise, we say that $\lambda \in \sigma_c(X, T)$. Below we also use the notation X' and T' to indicate respectively the topological dual space and the dual operator associated to (X, T) .

Lemma 2.5. ([2]) *Let X be a Banach separable space on \mathbb{C} and $T : X \rightarrow X$ be a linear bounded operator on X .*

- (1) *If T is hypercyclic then every connected component of the spectrum intersects the unit circle.*
- (2) *If T is hypercyclic, then $\sigma_{pt}(X', T') = \emptyset$,*
- (3) *If T is supercyclic then there exists a real number $R \geq 0$ such every connected component of the spectrum intersects the circle $\{z \in \mathbb{C}, |z| = R\}$.*
- (4) *If T is supercyclic, then $\sigma_{pt}(X', T')$ contains at most one point.*

In this paper, we will use only items 2) and 4) of Lemma 2.5. 1) and 3) are used in [1] for the study of dynamical properties of Markov chains associated to stochastic adding machines.

Remark 2.1. *If T is not supercyclic then λT is not hypercyclic for every fixed λ . However, it is possible to have T supercyclic and λT not hypercyclic for all sufficiently large (but fixed) λ .*

3. DYNAMICAL PROPERTIES OF MARKOV CHAINS OPERATORS

Let $Y = (Y_n)_{n \geq 1}$ be a discrete time irreducible Markov chain with countable state space E and with transition operator $A = [A_{i,j}]_{i,j \in E}$ (irreducible means that for each pair $i, j \in E$ there exists a nonnegative integer n such that $A_{i,j}^n > 0$). The Markov chain Y is said to be recurrent if the probability of visiting any given state is equal to one, otherwise Y is said to be transient. The Markov chain Y is called positive recurrent if it has an invariant probability distribution, i.e., there exists $u \in l^1$ such that $uA = u$. Every positive recurrent Markov chain is recurrent. If Y is recurrent but not positive recurrent, it is called null recurrent.

For the transient and null recurrent cases we have the following well-known equivalent definitions (see [9]):

- (i) A is transient if and only if $\sum_{n=1}^{+\infty} A_{i,j}^n < \infty$ for all $i, j \in E$.
- (ii) A is null recurrent if and only if $\lim_{n \rightarrow \infty} A_{i,j}^n = 0$ and $\sum_{n=1}^{+\infty} A_{i,j}^n = \infty$ for all $i, j \in E$.

Proposition 3.1. *Let A be a transition operator on an irreducible Markov chain with countable state space acting on c , then A is not supercyclic. The result remains valid if we replace c by c_0 in the positive recurrent case.*

Proof: Consider an enumeration of the state space so that we can consider $E = \mathbb{N}$ and the stochastic matrix $A = [A_{ij}]_{i,j \in \mathbb{N}}$ associated with the transition operator A . Assume that A is transient or null recurrent, then $\lim_{n \rightarrow \infty} A_{i,j}^n = 0$ for every i and j .

Now fix $y \in c - c_0$ (we do not need to consider the case $y \in c_0$ while considering density of orbits of y under A or λA because c_0 is a closed invariant subspace). Suppose that $\lim_{n \rightarrow \infty} y_n = \alpha \in \mathbb{C} - \{0\}$. We have that

$$\lim_{n \rightarrow \infty} (A^n y)_i = \alpha, \quad (3.1)$$

for every $i \in \mathbb{N}$. Indeed, since $\sum_{j=1}^{+\infty} A_{i,j}^n = 1$, for every $n \in \mathbb{N}$

$$\left| (A^n y)_i - \alpha \right| = \left| \sum_{j=1}^{+\infty} A_{i,j}^n (y_j - \alpha) \right| \leq (\|y\|_\infty + |\alpha|) \sum_{j=1}^m A_{i,j}^n + \sup_{j \geq m+1} |y_j - \alpha|.$$

The second term in the rightmost side of the previous expression can be made arbitrarily small by choosing m sufficiently large while the first one goes to zero as n tends to $+\infty$ for every choice of m . Hence (3.1) holds.

From (3.1), we have that

$$\liminf_{n \rightarrow \infty} \frac{|(A^n y)_i|}{\|A^n y\|_\infty} \geq \frac{|\alpha|}{\|y\|_\infty} > 0,$$

and then $\{(A^n y) / \|A^n y\|_\infty : n \geq 1\}$ is not dense in the unit sphere of c centered at 0 which implies that $\{\lambda A^n y : \lambda \in \mathbb{C}, n \geq 1\}$ is not a dense subset of c . Since y is arbitrary, A is not supercyclic on c .

Now, assume that A is positive recurrent, then there exists an invariant measure $u \in l^1 \setminus \{0\}$ such that $uA = u$, hence $uA^n = u$ for all integer $n \geq 1$.

Suppose that A is supercyclic. Take $y \in c \cap S_c^1$, where $S_c^1 = \{x \in c : \|x\|_\infty = 1\}$ such that projective orbit of y under A is dense in S_c^1 , then for all $x \in S_c^1$, there exists an increasing sequence $(n_k)_{k \geq 0}$ such that $\lim_{k \rightarrow \infty} \frac{A^{n_k} y}{\|A^{n_k} y\|_\infty} = x$. Since $\|A^{n_k} y\|_\infty \leq \|y\|_\infty = 1$, then

$$|\langle u, x \rangle| = \lim_k \frac{|\langle u, A^{n_k} y \rangle|}{\|A^{n_k} y\|_\infty} = \lim_k \frac{|\langle u, y \rangle|}{\|A^{n_k} y\|_\infty} \geq |\langle u, y \rangle|.$$

Where $\langle u, z \rangle$ is the scalar product between u and z for z in c . Since x is arbitrary in S_c^1 , the last inequality implies that $u = 0$ and this is an absurd. Hence the projective orbit of y under A could not be dense in S_c^1 which means that A is not supercyclic.

To finish the proof we just point out that in the positive recurrent case, the same proof holds if we replace c by c_0 . \square

Another result is:

Proposition 3.2. *Let $q \geq 1$ and $A : l^q \rightarrow l^q$ be an hypercyclic (supercyclic) operator on l^q , then*

- (1) *If $A(c_0) \subset c_0$, then A is also hypercyclic (supercyclic) on c_0 .*
- (2) *If $r > q$ and $A(l^r) \subset l^r$, then A is also hypercyclic (supercyclic) on l^r .*

Proof: (1) Suppose that A is hypercyclic on l^q and let $x \in l^q$ be a hypercyclic vector, i.e $\overline{O(x)} = l^q$. Now fix $y \in c_0$ and $\epsilon > 0$. Take $m \in \mathbb{N}$ such that $\sup_{i > m} |y_i| \leq \epsilon/2$. Define $y^{(m)}$ as

$$y_i^{(m)} = \begin{cases} y_i & , 1 \leq i \leq m, \\ 0 & , \text{otherwise,} \end{cases}$$

for every $i \in \mathbb{N}$. Since $y^{(m)} \in l^q$, there exists $n \in \mathbb{N}$ such that $\|A^n x - y^{(m)}\|_\infty \leq \|A^n x - y^{(m)}\|_q \leq \epsilon/2$. Therefore

$$\|A^n x - y\|_\infty \leq \|A^n x - y^{(m)}\|_\infty + \|y^{(m)} - y\|_\infty \leq \epsilon.$$

Since ϵ and y are arbitrary, x is a hypercyclic vector in c_0 .

The proof in the supercyclic case is analogous.

(2) The proof is analogous to item 1) and come from the fact that if $1 \leq q < r$, then $l^q \subset l^r$. \square

Corollary 3.3. *Let $A : X \rightarrow X$ where $X \in \{l^q, q \geq 1\}$ be the transition operator of a irreducible positive recurrent stochastic Markov chain, then A is not supercyclic.*

Proof: From Proposition 3.1 we have that A acting on c_0 is not supercyclic. Thus from (1) in Proposition 3.2 we obtain that A acting on X is not supercyclic. \square

Remark 3.1. *Since an operator A is supercyclic if and only if cA is supercyclic for $c \neq 0$, then all the previous results in this section hold for operators associated to countable non-negative irreducible matrices with each line having the same sum of their entries.*

Question: What happens if $A : X \rightarrow X$ is a countable infinite non-negative irreducible matrix where the the sum of entries of lines is not constant?

Is A not supercyclic on c ?

If A is positive recurrent (see [8] for the definition), Can we prove that A is not supercyclic in $X \in \{c_0, c, l^q, q \geq 1\}$?

4. SIMPLE RANDOM WALKS

Consider the nearest neighbor simple random walk on \mathbb{Z}_+ with partial reflection at the boundary and jump probability $p \in (0, 1)$ (when at zero, the walk stays at zero with probability $1 - p$). Denote by $W_p := W = (W_{i,j})_{i,j \geq 0}$ its transition operator. We have $W_{0,0} = 1 - p$, $W_{0,1} = p$ and for all $i \geq 1$, $W_{i,j} = 0$ if $j \notin \{i - 1, i + 1\}$, $W_{i,i-1} = 1 - p$, $W_{i,i+1} = p$ for all $i \geq 1$. We have

$$W_p = \begin{bmatrix} 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1-p & 0 & p & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1-p & 0 & p & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is known ([9]) that the simple random walk on \mathbb{Z}_+ is positive recurrent if $p < 1/2$, null recurrent if $p = 1/2$ and transient if $p > 1/2$.

In particular, from Proposition 3.1 and Corollary 3.3, we have that W_p acting on $X \in \{c_0, c, l^q, q \geq 1\}$ is not supercyclic if $p < 1/2$.

Theorem 4.1. *Let $X \in \{c_0, l^q, q \geq 1\}$. If $p > 1/2$, then the infinite matrix W_p of the simple random walk on \mathbb{Z}_+ is supercyclic on X . Moreover λW_p is frequently hypercyclic and chaotic for all $|\lambda| > \frac{1}{2p-1}$.*

Before we prove Theorem 4.1 we need three technical results:

Lemma 4.2. *Let $X \in \{c_0, l^q, q \geq 1\}$, then $\sigma_{pt}(X, W_p)$ is not empty if and only if $p > 1/2$, moreover, in this case $0 \in \sigma_{pt}(X, W_p)$.*

Proof: Let λ be an element of $\sigma_{pt}(W_p)$ and $u = (u_n)_{n \geq 0}$ be an eigenvector associated to λ , then

$$(1 - p - \lambda)u_0 + pu_1 = 0, (1 - p)u_n - \lambda u_{n+1} + pu_{n+2} = 0, \forall n \geq 0.$$

We deduce that there exists a sequence of complex numbers $(q_n)_{n \geq 0}$ where $q_0 = 1$, $q_1 = \frac{\lambda + p - 1}{p}$ and $u_n = q_n u_0$. Moreover

$$\begin{bmatrix} q_n \\ q_{n-1} \end{bmatrix} = M \begin{bmatrix} q_{n-1} \\ q_{n-2} \end{bmatrix}, \quad \forall n \geq 2.$$

Where $M = \begin{bmatrix} \frac{\lambda}{p} & \frac{p-1}{p} \\ 1 & 0 \end{bmatrix}$. Hence $\begin{bmatrix} q_n \\ q_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} q_1 \\ q_0 \end{bmatrix}$ for all $n \geq 2$. Assume that $\lambda^2 \neq 4p(1 - p)$, then the matrix M have distinct eigenvalues and hence it is diagonalizable. Therefore, there exist $c, d \in \mathbb{C} \setminus \{0\}$ such that $q_n = c\alpha^n + d\beta^n$ for all integer $n \geq 0$, where α, β are the eigenvalues of M . Since $\alpha\beta = \det(M) = \frac{1-p}{p}$, then if $0 < p < 1/2$, we have $\alpha\beta > 1$. Then either $|\alpha| > 1$ or $|\beta| > 1$, Hence $(q_n)_{n \geq 0}$ is not bounded and therefore the point spectrum of W_p is empty.

If $p = 1/2$, then either ($|\alpha| > 1$ or $|\beta| > 1$) or α, β are conjugated complex numbers of modulus = 1. In both cases the point spectrum of W_p is empty.

If $\lambda^2 = 4p(1-p)$, then the matrix M is not diagonalisable and has a unique eigenvalue θ . In this case, there exist $e, f \in \mathbb{C} \setminus \{0\}$ such that $q_n = (e + fn)\theta^n$ for all integer $n \geq 0$. If $p \leq 1/2$, then $|\theta| = \sqrt{\frac{1-p}{p}} \geq 1$, hence q_n is not bounded. We deduce that the point spectrum of W_p is empty.

Now assume that $1/2 < p < 1$ and $\lambda = 0$ (M diagonalizable), then $\alpha, \beta \in \{-\sqrt{\frac{1-p}{p}}i, \sqrt{\frac{1-p}{p}}i\}$, therefore α, β are complex conjugated numbers of modulus < 1 . Hence $0 \in \sigma_{pt}(X, W_p)$. If $p = 1$ and $\lambda = 0$, then $q_0 = e$ and $q_n = 0$ for all $n \geq 1$. Thus $0 \in \sigma_{pt}(X, W_p)$. \square

Remark 4.1. Consider M as in the proof of Lemma 4.2. 1. Since the eigenvalues of M depend continuously of λ , we deduce that for all $p > 1/2$, there exists $0 < r_p < 1$ such that $D(0, r_p) \subset \sigma_{pt}(X, W_p)$. In particular, we can prove that $[0, 2\sqrt{1-p}) \subset \sigma_{pt}(X, W_p)$.

2. If $p = 1/2$ the eigenvalues of M are $\lambda \pm \sqrt{\lambda^2 - 1}$, we deduce that if $X = l^\infty$, the interval $] -1, 1[\subset \sigma_{pt}(l^\infty, W_p)$.

Lemma 4.3. Let $X \in \{c_0, l^q, q \geq 1\}$ and $v = (v_i)_{i \geq 0} \in X$. Let $a = (a_n)_{n \geq 0} \in l^1$ and $x = (x_n)_{n \geq 0}$ defined by

$$x_n = \sum_{k=0}^n a_k v_{n-k}, \quad \forall n \in \mathbb{Z}_+.$$

Then $(x_n)_{n \geq 0} \in X$, moreover

$$\|x\| \leq \|a\|_1 \|v\|.$$

Proof: By putting $v_k = 0$ for all $k < -1$, we can assume that $x_n = \sum_{k=0}^{+\infty} a_k v_{n-k}$ for all $n \geq 0$.

Now suppose that $X = c_0$. For each $i \in \mathbb{N}$

$$|x_n| \leq \left(\sum_{k=0}^i |a_k| \right) \sup_{0 \leq k \leq i} \|v_{n-k}\| + \left(\sum_{k=i+1}^{\infty} |a_k| \right) \|v\|_\infty.$$

Since $(v_n)_{n \geq 0} \in c_0$ and $(a_n)_{n \geq 0} \in l^1$, we deduce that $(x_n)_{n \geq 0} \in c_0$.

If $X = l^1$, we have

$$\sum_{n=0}^{+\infty} |x_n| \leq \sum_{n=0}^{+\infty} \sum_{k=0}^{\infty} |a_k v_{n-k}| \leq \left(\sum_{k=0}^{\infty} |a_k| \right) \left(\sum_{n=0}^{+\infty} |v_n| \right) = \|a\|_1 \|v\|_1.$$

Now, assume that $X = l^q$, $1 < q < \infty$, we consider its conjugate exponent r (i.e., $1/q + 1/r = 1$). We have $|x_n| \leq \sum_{k=0}^{\infty} |a_k|^{1/r} |a_k|^{(1-1/r)} |v_{n-k}|$. Hence By Hölder's inequality, we obtain

$$|x_n| \leq \left(\sum_{k=0}^{\infty} |a_k| \right)^{\frac{1}{r}} \left(\sum_{k=0}^{\infty} |a_k|^{q(1-1/r)} |v_{n-k}|^q \right)^{\frac{1}{q}}.$$

As a consequence, $\sum_{n=0}^{+\infty} |x_n|^q \leq \left(\sum_{k=0}^{\infty} |a_k| \right)^{q-1} \sum_{n=0}^{+\infty} \sum_{k=0}^{\infty} |a_k| |v_{n-k}|^q$. Hence

$$\sum_{n=0}^{+\infty} |x_n|^q \leq \left(\sum_{k=0}^{\infty} |a_k| \right)^q \sum_{n=0}^{+\infty} |v_n|^q.$$

□

Proposition 4.4. *Let $X \in \{c_0, l^q, q \geq 1\}$ and $p > 1/2$ then for $v = (v_i)_{i \geq 0} \in X$ there exists $u = (u_i)_{i \geq 0} \in X$ such that $W(u) = v$. Moreover, u is explicitly given by*

$$u_n = \frac{1}{p} \left(\sum_{j=0}^{\lfloor n/2 \rfloor} \left(\frac{p-1}{p} \right)^j v_{n-2j-1} + \left(\frac{p-1}{p} \right)^{\lfloor n/2 \rfloor + 1} p u_0 \right) \quad (4.1)$$

where $\lfloor n/2 \rfloor$ is the largest integer $< n/2$.

Proof: Fix $v = (v_i)_{i \geq 0} \in X$ and $u = (u_i)_{i \geq 0} \in l^\infty$ such that $W(u) = v$, then

$$u_1 = \frac{v_0}{p} + \frac{p-1}{p} u_0, \quad u_n = \frac{v_{n-1}}{p} + \frac{p-1}{p} u_{n-2}, \quad \forall n \geq 2. \quad (4.2)$$

Then we obtain by induction that for all integer $n \geq 2$,

$$u_n = 1/p (v_{n-1} + \gamma v_{n-3} + \gamma^2 v_{n-5} + \dots + \gamma^k v_{n-2k-1} + \dots + \gamma^{\lfloor n/2 \rfloor} v_t + \gamma^{\lfloor n/2 \rfloor + 1} p u_0),$$

where $\gamma = \frac{p-1}{p}$, $t = 0$ if n is odd and $t = 1$ otherwise. Thus we have (4.1).

By Lemma 4.3, we deduce that u belongs to X .

Observe that if $u \in c_0$, then

$$\|u\|_\infty \leq \delta \max(\|v\|_\infty, |u_0|) \quad \text{where } \delta = \frac{1}{p} \sum_{n=0}^{+\infty} \gamma^n = \frac{1}{2p-1}. \quad (4.3)$$

If $u \in l^q$, $q \geq 1$, then by Lemma 4.3,

$$\|u\|_q \leq \frac{1}{2p-1} \|v\|_q + |u_0| \left(\sum_{n=1}^{+\infty} \gamma^{nq} \right)^{1/q}. \quad (4.4)$$

Remark 4.2. *If $u_0 = 0$, then $\|u\|_q \leq \frac{1}{2p-1} \|v\|$ (in X).*

Lemma 4.5. *Let $X \in \{c_0, l^q, q \geq 1\}$, $p > 1/2$, then $D = \bigcup_{n=1}^{+\infty} \text{Ker}(W^n)$ is dense in X .*

Proof: Fix X in $\{c_0, l^q, q \geq 1\}$. By Proposition 4.2, we have that $0 \in \sigma_{pt}(X, W)$. Then for all integer $n \geq 1$, $\text{Ker}(W^n)$ is not empty.

Claim: *for all integer $n \geq 1$, there exists $V_{0,n}, \dots, V_{n-1,n} \in l^\infty$, linearly independent such that if $u = (u_i)_{i \geq 0} \in \text{Ker}(W^n)$, then $u = \sum_{i=0}^{n-1} u_i V_{i,n}$.*

Indeed, since $W_{j,k} = 0$ for all $k \geq j+2$, we deduce that for all integer $n \geq 2$, $W_{j,k}^n = 0$ for all integer $k \geq j+n+1$.

Assume that $u = (u_i)_{i \geq 0} \in \text{Ker}(W^n)$. The relation $\sum_{k=0}^{j+n} W_{j,k}^n u_k = 0$ holds for all $j \in \mathbb{N}$. Since $W_{0,n}^n > 0$, we deduce that

$$u_n = \sum_{i=0}^{n-1} u_i c_{i,n,n},$$

where $c_{i,n,n} = -\frac{W_{0,i}^n}{W_{0,n}^n}$ for all $i = 0, \dots, n-1$. We also obtain by induction that $u_k = \sum_{i=0}^{n-1} u_i c_{i,k,n}$ for all $k \geq n$, where $c_{i,k,n}$ are real numbers.

For all $i \in \{0, 1, \dots, n-1\}$, define the infinite vector $V_{i,n} = (V_{i,n}(k))_{k \geq 0}$ by putting $V_{i,n}(k) = c_{i,k,n}$ for all $k \geq n$ and $V_{i,n}(k) = \delta_{i,k}$ for all $0 \leq k < n$. Then, we obtain the claim.

Now observe that $V_{i,n} \in X$ for every integer n and $i = 0, \dots, n$. Indeed for all $i = 0, \dots, n$, we have $W^{n-1}V_{i,n} \in \text{ker}W$ that is contained in X . Hence by Lemma 4.4, we deduce that $W^{n-2}V_{i,n} \in X$ and continuing by the same way, we obtain that $V_{i,n} \in X$.

Now, let $z = (z_i)_{i \geq 0} \in X$, such that $z_i = 0$ for all $i > n$ where n is a large integer number, then z can be approximated by the vector $\sum_{i=0}^{n-1} z_i V_{i,n}$ which belongs to the set D . Hence the D is dense in X . \square

Proof of Theorem 4.1: First we prove that W is supercyclic. Recall the definition of D from the statement of Lemma 4.5. By Proposition 4.4, for every $v \in D$, we can choose $Sv \in D$ such that $W(Sv) = v$. Using the fact that $Sv \in D$, we prove by induction that $W^n(S^n v) = v$ for all $v \in D$. On the other hand, since for all $u \in D$, there exists a nonnegative integer N such that $W^n(u) = 0$ for all $n \geq N$, we deduce that $\lim \|W^n(u)\| \|S^n(v)\| = 0$, $\forall u, v \in D$. Hence W satisfies the supercyclicity criterion. Thus W is supercyclic.

Claim: λW is frequently hypercyclic and chaotic for all $|\lambda| > \frac{1}{2^{p-1}}$.

Indeed, let $v = (v_i)_{i \geq 0} \in X$ and $u = (u_i)_{i \geq 0} \in l^\infty$ such that $W(u) = v$, then u satisfies (4.1). Putting $u_0 = 0$, we obtain $S(v) = (0, u_1, u_2, \dots)$ and $W(Sv) = v$.

We also have by remark 4.2 that $\|Sv\| \leq \frac{1}{2^{p-1}} \|v\|$.

On the other hand, since $S(v)_0 = 0$, we obtain by (4.2) that

$$S^2(v) = (0, 0, (S^2v)_2, (S^2v)_3, \dots).$$

We deduce that for all integer $n \geq 0$

$$S^n(v) = (\underbrace{0, \dots, 0}_n, (S^n v)_n, (S^n v)_{n+1}, \dots) \text{ and } \|S^n(v)\| \leq \left(\frac{1}{2^{p-1}}\right)^n \|v\|.$$

Let λ be a complex number such that $|\lambda| > \delta$, then $\|\lambda^{-n} S^n(v)\|$ converges to 0 exponentially as n goes to $+\infty$.

Taking $W' = \lambda W$ and $S' = \lambda^{-1} S$ and $D = \cup_{n=0}^{+\infty} \text{Ker}(W^n)$, we obtain that the series $\sum_{n=0}^{+\infty} W'^n(x)$ and $\sum_{n=0}^{+\infty} S'^n(x)$ are absolutely convergent and hence unconditionally convergent for all $x \in D$, moreover $W' \circ S' = I$ on D , then we are done by Theorem 2.4. \square

It is known (see chapter 5 in [9]) that G_p is transient if and only if

$$S_1 = \sum_{n=1}^{+\infty} \frac{(1-p_1)(1-p_2)\dots(1-p_n)}{p_1 p_2 \dots p_n} < \infty,$$

and positive recurrent if and only if

$$S_2 = \sum_{n=1}^{+\infty} \frac{p_0 p_1 \dots p_{n-1}}{(1-p_1)(1-p_2)\dots(1-p_n)} < \infty.$$

Thus if both series S_1 and S_2 do not converge, then G is null recurrent.

Now, consider the sequence

$$w_n = \frac{(1-p_1)(1-p_3)\dots(1-p_{n-1})}{p_1 p_3 \dots p_{n-1}} \text{ for } n \text{ even,}$$

and

$$w_n = \frac{(1-p_0)(1-p_2)\dots(1-p_{n-3})(1-p_{n-1})}{p_0 p_2 \dots p_{n-3} p_{n-1}} \text{ for } n \text{ odd.}$$

Theorem 4.8. *The following properties hold:*

1. *If $X = c_0$ and $\lim w_n = 0$ or $X = l^q$, $q \geq 1$ and $\sum_{n=1}^{+\infty} w_n^q < +\infty$, then G is supercyclic on X .*

2. *Let $X \in \{c_0, l^q, q \geq 1\}$ and assume that there exist $n_0 \in \mathbb{N}$ and $\alpha > 0$ such that $p_n \geq \frac{1}{2} + \alpha$ for all $n \geq n_0$, then there exists $\delta > 1$ such that λG is frequently hypercyclic and chaotic for all $|\lambda| > \delta$.*

Remark 4.4. *Item 1 in Theorem 4.8 implies that there exist null recurrent random walks which are supercyclic on c_0 .*

Proof: 1. Assume that 0 is an eigenvalue of G associated to an eigenvector $u = (u_n)_{n \geq 0}$. Then

$$u_1 = \frac{p_0 - 1}{p_0} u_0 \text{ and } u_n = \frac{p_{n-1} - 1}{p_n} u_{n-2}, \forall n \geq 2.$$

Thus $u_n = (-1)^n w_n u_0$. If $X = c_0$ then $0 \in \sigma_{pt}(T)$ if and only if $\lim w_n = 0$. If $X = l^q$, $q \geq 1$, then $0 \in \sigma_{pt}(G)$ if and only if $\sum_{n=1}^{+\infty} w_n^q < +\infty$. In both cases, we deduce, exactly as done in the proof of Theorem 4.1, that G is supercyclic on X .

2. Assume that $\sum_{n=1}^{+\infty} w_n < +\infty$,

Indeed, let $v = (v_i)_{i \geq 0} \in X$ and $u = (u_i)_{i \geq 0} \in l^\infty$ such that $G(u) = v$ and $u_0 = 0$, then

$$u_1 = \frac{v_0}{p_0} \quad \text{and} \quad u_n = \frac{v_{n-1}}{p_{n-1}} + \frac{p_{n-1} - 1}{p_{n-1}} u_{n-2} \text{ for all integer } n \geq 2.$$

Putting $r_n = \frac{p_n - 1}{p_n}$ for all integer $n \geq 0$, we obtain by induction that for all integer $n \geq 2$,

$$u_n = \frac{1}{p_{n-1}} v_{n-1} + \frac{1}{p_{n-3}} r_{n-1} v_{n-3} + \frac{1}{p_{n-5}} r_{n-1} r_{n-3} v_{n-5} + \dots \\ + \frac{1}{p_{n-2k+1}} (r_{n-1} r_{n-3} \dots r_{n-2k+1}) v_{n-2k-1} + \dots + \frac{1}{p_t} (r_{n-1} r_{n-3} \dots r_{t+2}) v_t,$$

where $t = 0$ if n is odd and $t = 1$ otherwise. Put $S(v) = (0, u_1, \dots)$ for all $v \in X$, then $G(Sv) = v$. Since $p_n \geq \frac{1}{2} + \alpha$ for all $n \geq n_0$, we have that $r_n \leq \frac{1/2 - \alpha}{1/2 + \alpha} < 1$. We deduce exactly as done in the proof of Theorem 4.1 that there exists $\delta > 1$ such that λT is frequently hypercyclic and chaotic for all $|\lambda| > \delta$. \square

Questions: 1. If $X \in \{c_0, l^q, q \geq 1\}$ and $\sum_{n=1}^{+\infty} w_n < +\infty$, can we prove that there exists $\delta > 1$ such that λG is frequently hypercyclic and chaotic for all $|\lambda| > \delta$?

2. If $\sum_{n=1}^{+\infty} w_n < +\infty$, then by Hölder inequality, we deduce that

$$\sum_{n=1}^{+\infty} \frac{(1-p_1)(1-p_2)\dots(1-p_n)}{p_0 p_1 \dots p_{n-1}} < +\infty$$

and hence G is transient.

Does there exist G transient and not supercyclic on l^1 such that $\sum_{n=1}^{+\infty} w_n < +\infty$?

Theorem 4.9. *If $X = c_0$ and $\sum_{n=1}^{+\infty} w_n^{-1} < +\infty$ or $X = l^1$ and $1/w_n$ is bounded or $X = l^q, q > 1$ and $\sum_{n=1}^{+\infty} (w_n)^{-\frac{q}{q-1}} < +\infty$, then λG is not hypercyclic for all $|\lambda| > 1$.*

Remark 4.5. *Theorem 4.9 is the closest we get to Theorem 4.6. We conjecture that G is not supercyclic under the hypothesis of Theorem 4.9.*

Proof: Assume that 0 is an element of $\sigma_{pt}(X', G')$ and $u = (u_n)_{n \geq 0}$ an eigenvector associated to 0, then $uT = 0$. Thus $(1-p_0)u_0 + (1-p_1)u_1 = 0$ and $p_n u_n + (1-p_{n+2})u_{n+2} = 0$ for all $n \geq 0$. Hence for all $n \geq 1$, we have

$$u_{2n} = \frac{p_{2n-2} p_{2n-4} \dots p_0}{(p_{2n-2} - 1)(p_{2n-4} - 1) \dots (p_0 - 1)} u_0$$

and

$$u_{2n+1} = \frac{p_{2n-1} p_{2n-3} \dots p_1}{(p_{2n-1} - 1)(p_{2n-3} - 1) \dots (p_1 - 1)} \frac{(1-p_0)}{p_1 - 1} u_0.$$

Hence if $X = c_0$ and $\sum_{n=1}^{+\infty} w_n^{-1} < +\infty$ or $X = l^1$ and $1/w_n$ is bounded or $X = l^q, q > 1$ and $\sum_{n=1}^{+\infty} (w_n)^{-\frac{q}{q-1}} < +\infty$, we have $0 \in \sigma_{pt}((\lambda G)', X')$. Thus, by Lemma 2.5, λG is not hypercyclic for all λ . \square

Question: If G is null recurrent and $X = l^1$, Is λT is not hypercyclic for all $|\lambda| > 1$?

Remark 4.6. *Let $\bar{p} = (p_n)_{n \in \mathbb{Z}}$ be a sequence of probabilities, and denote by $\bar{G}_{\bar{p}} := \bar{G}$ the transition operator of the spatially inhomogeneous simple random walks on \mathbb{Z} , defined by: For all $i \in \mathbb{Z}$, $\bar{G}_{i,i-1} = 1 - p_i$, $\bar{G}_{i,i+1} = p_i$ and $\bar{G}_{i,j} = 0$ if $j \notin \{i-1, i+1\}$. Then by using the same method done in Theorem 4.8, we can prove the following results:*

1. *If $\lim w_n = 0$ and $\lim w_{-n}^{-1} = 0$ then \bar{G} is supercyclic on c_0 . If $q \geq 1$ and $\sum_{n=1}^{+\infty} w_n^q < +\infty$ and $\sum_{n=1}^{+\infty} w_{-n}^{-q} < +\infty$, then \bar{G} is supercyclic on l^q .*

2. *Let $X \in \{c_0, l^q, q \geq 1\}$ and assume that there exist positive constants n_0, n_1, α such that $p_n \leq \frac{1}{2} - \alpha$ for all $n \geq n_0$ and $p_n \geq \frac{1}{2} + \alpha$ for all $n \leq -n_1$,*

then there exists $\delta > 1$ such that $\lambda\bar{G}$ is frequently hypercyclic and chaotic for all $|\lambda| > \delta$.

Question: Does there exist a transient Markov operator that is not supercyclic? Does there exist a null recurrent Markov operator that is supercyclic on l^1 ?

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